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SLOW DECAY RADIAL SOLUTIONS TO AN ELLIPTIC EQUATION WITH SUB AND SUPERCRITICAL EXPONENTS

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ABSTRACT. We study radial solutions of the problem

$$\Delta u + u^p + u^q = 0, \ u > 0 \quad \text{in } \mathbb{R}^N,$$

where $\frac{N}{N-2} < p < \frac{N+2}{N-2} < q$ and $N \geq 3$. We show that if p is close to $\frac{N}{N-2}$, q is close to $\frac{N+2}{N-2}$, and a certain relation holds between them, then the problem has slowly decaying solutions.

1. INTRODUCTION

Let $N \geq 3$. We are interested in finding radially symmetric solutions $u(r)$, $r = |x|$, to

$$\Delta u + u^p + u^q = 0, \ u > 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where

$$\frac{N}{N-2} < p < \frac{N+2}{N-2} < q. \tag{1.2}$$

Solutions of (1.1) such that

$$\lim_{|x|\to+\infty} u(x) = 0$$

are called ground states. A ground state such that

$$\lim_{|x|\to+\infty} |x|^{N-2} u(x)$$

exists and is positive is said to have fast decay, and if it satisfies

$$\lim_{|x|\to+\infty} |x|^{\frac{2}{p-1}} u(x) = \ell > 0$$

we say that u has slow decay. In this case the constant ℓ depends on p and N only and is given by

$$K_p := \left(\frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) \right)^{\frac{1}{p-1}}.$$

When $p = q$ in equation (1.1) the existence of ground states is well understood. If $p < \frac{N+2}{N-2}$ there are no solutions of (1.1) [6], while if $p \geq \frac{N+2}{N-2}$ ground states do exist. In the case of the critical exponent $p = \frac{N+2}{N-2}$ all solutions are necessarily radial around some point [5]. Radial ground states in the critical or supercritical case are parametrized by $u(0)$ and are unique, up to the natural scaling of the equation. In the critical case the ground state is explicit and has fast decay, while in the supercritical case the radial ground state has slow decay.

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Lin and Ni considered equation (1.1) in [9] to provide a counter example to the Nodal Domain Conjecture, and found slow decay ground states of (1.1) when $q = 2p - 1$ given explicitly by

$$u(x) = a(b + |x|^2)^{-\frac{2}{q-1}} = a(b + |x|^2)^{-\frac{1}{p-1}} \tag{1.3}$$

with $a = K_p$ and $b = \frac{1}{p}\left(N - 2 - \frac{2}{p-1}\right)^2$.

W.-M. Ni then asked whether there are radial ground states of (1.1) under condition (1.2). In their work Bamón, Flores, and del Pino [1] addressed this question and discovered a complex picture of solutions. First, they found an increasing number of fast decay ground states, if one of the exponents is fixed and the other one is sufficiently close to $\frac{N+2}{N-2}$. More precisely, they proved that for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ fixed, given any integer $k \geq 1$, if $q > \frac{N+2}{N-2}$ is close enough to $\frac{N+2}{N-2}$ then (1.1) has at least k radial ground states with fast decay. They also showed that if $\frac{N+2}{N-2} < q$ is fixed, given any integer $k \geq 1$, if $p < \frac{N+2}{N-2}$ is sufficiently close to $\frac{N+2}{N-2}$ then (1.1) has at least k radial ground states with fast decay. Furthermore, if $q > \frac{N+2}{N-2}$ is fixed there exists $p_0 > \frac{N}{N-2}$ such if $1 < p < p_0$ then there are no radial ground states. The authors in [1] obtain their results using dynamical systems arguments. Recently Campos [2] gave a different proof of the same main result.

Our main interest in this work is the existence of slow decay radial solutions. These solutions are unique if they exist. Indeed, following [1, p. 555] (see also [7]), after an Emden-Fowler change of variables, equation (1.1) is transformed into a first order 3 dimensional system of ODE. Slow decay solutions correspond to trajectories contained in the 1 dimensional stable manifold of a stationary point, which implies the uniqueness. But regular slow decay solutions must also lie in the 2 dimensional unstable manifold of another stationary point, which suggests that their existence is non-generic in the parameters p, q.

The only indication of existence of regular slow decay radial solutions is a result in [1], where it is proved that if $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ is fixed there is a sequence $q_j > \frac{N+2}{N-2}$, $q_j \to \frac{N+2}{N-2}$ such that for these exponents there is a radial solution with slow decay, but it is unknown whether it is regular or singular.

We conjecture that *slow decay singular solutions* do not exist or exist for a finite choices (p, q), since they must satisfy two constraints

$$\lim_{|x|\to 0} |x|^{\frac{2}{q-1}} u(x) = K_q \quad \text{and} \quad \lim_{|x|\to\infty} |x|^{\frac{2}{p-1}} u(x) = K_p.$$

If one can discard the existence of slow decay singular solutions, the result of [1] would imply the existence of slow decay radial regular solutions associated to the sequence of exponents q_j.

In this work we prove the existence of slow decay radial regular solutions when p is close to $\frac{N}{N-2}$, q is close to $\frac{N+2}{N-2}$ and both are related by some equation. More precisely, let $\varepsilon > 0$, $\delta > 0$ and assume

$$p = \frac{N}{N-2} + \varepsilon, \quad q = \frac{N+2}{N-2} + \delta \tag{1.4}$$

with $\varepsilon = O(\delta)$.

Theorem 1.1. *Let $k \geq 2$ be an integer. Then there exists $\delta_0(k) > 0$ and a function $\varepsilon_k(\delta) > 0$ such that for $0 < \delta \leq \delta_0(k)$ and $\varepsilon = \varepsilon_k(\delta)$ there exists a radial slow decay*

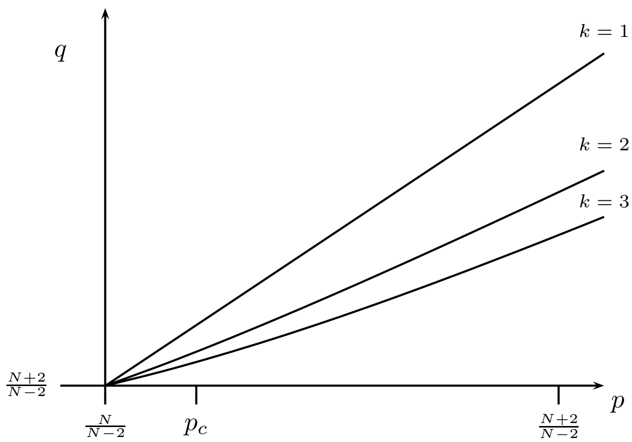

FIGURE 1. Bifurcation diagram for (1.1) showing regular slow decay solutions for $N = 5$.

solution u of (1.1) with exponents given by (1.4). Moreover, there exist constants α_j, $j = 1 \dots k$ which depend on N and k, and the solution u has the form:

$$u(x) = \gamma_N \sum_{j=1}^{k} \left(\frac{1}{1 + (\alpha_j \delta^{-(j+\frac{N-4}{2})})^{\frac{4}{N-2}} |x|^2} \right)^{\frac{1}{p-1}} \delta^{-(j+\frac{N-4}{2})} \alpha_j \ (1 + o(1)) \ ,$$

and $\varepsilon_k(\delta)$ satisfies

$$\varepsilon_k(\delta) = \frac{k}{2}\delta + o(\delta) \qquad \text{as } \delta \to 0.$$

where $\gamma_N = (N(N-2))^{\frac{N-2}{4}}$, and $o(1) \to 0$ uniformly on $\mathbb{R}^N$ as $\delta \to 0$.

The constants $\alpha_1, \dots, \alpha_k$ have explicit formulas in terms of the numbers Λ_j^* given in (2.8), from which it follows that

$$\alpha_1 = \lim_{\delta \to 0} \frac{\gamma_N}{K_p} \delta^{\frac{1}{p-1}}.$$

This is consistent with the behavior of the slow decay solutions

$$\lim_{|x| \to \infty} |x|^{\frac{2}{p-1}} u(x) = K_p.$$

Solutions to (1.1) corresponding to $k = 1$ are the explicit ones found by Lin and Ni and given in (1.3). In this case $q = 2p - 1$ which corresponds to the relation $\delta = 2\varepsilon$. We believe that the solutions we construct in Theorem 1.1 are the same as the ones detected in [1] when p is close to $\frac{N}{N-2}$ and q is close to $\frac{N+2}{N-2}$.

The existence of slow decay solutions is interesting due to the following result of Flores [7]. If for some p, q in the range (1.2) there is a radial ground state with slow decay and also

$$p > p_c := \frac{N + 2\sqrt{N-1}}{N + 2\sqrt{N-1} - 4}$$

then there are infinitely many radial ground states with fast decay.

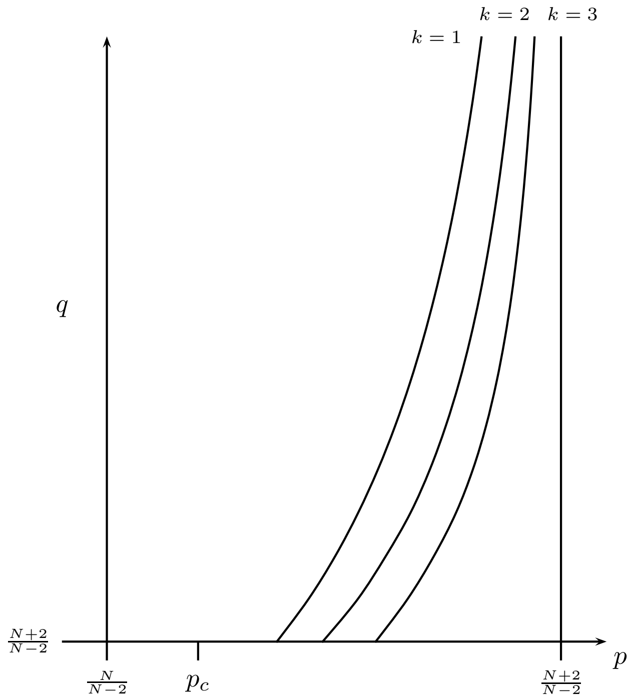

FIGURE 2. Bifurcation diagram for (1.1) showing singular fast decay solutions for $N = 5$.

In Figure 1 we show a bifurcation diagram for (1.1) based on numerical computations. In the axes we have the values of p and q, horizontal and vertical respectively, for which we have found numerically a regular radial slow decay solution. For p close to $\frac{N}{N-2}$ and q close to $\frac{N+2}{N-2}$ we think that these solutions are exactly the ones constructed in Theorem 1.1 for curves $q = q_k(p)$, $k = 1, 2, 3, \ldots$. For $k = 1$ the curve is the line $q = 2p - 1$ and for $k = 2$ and 3 the curves start at $p = \frac{N}{N-2}$ with a derivative consistent with Theorem 1.1, and then they slightly bend upwards. The numerical computations show that these curves can be continued even for $p > \frac{N+2}{N-2}$. Hence, at least numerically, we see that solutions with slow decay exist for $p > p_c$ and $q = q_k(p)$ and therefore the result of Flores [7] applies.

A dual phenomenon to the existence of bounded solutions with slow decay is the existence of singular solutions with fast decay. In [1], the authors showed that if $q > \frac{N+2}{N-2}$ is fixed there is a sequence $p_j < \frac{N+2}{N-2}$, $p_j \to \frac{N+2}{N-2}$ such that for these exponents there is either a fast decay singular solution or a slow decay singular solution.

Numerically we found a family of curves relating $p \in (\frac{N}{N-2}, \frac{N+2}{N-2})$ and $q > \frac{N+2}{N-2}$ for which singular fast decay solutions exist, see Figure 2. These curves are asymptotic to the line $p = \frac{N+2}{N-2}$ as $q \to \infty$.

Noting that singular solutions satisfy

$$\lim_{|x| \to 0} |x|^{\frac{2}{q-1}} u(x) = K_q,$$

and using formal asymptotic expansions we arrive at the following conjecture.

Conjecture 1.2. *Let $k \geq 1$ be an integer. Assume*

$$p = \frac{N+2}{N-2} - \varepsilon.$$

Then there exists $\varepsilon_0 > 0$ and a function $q_k(\varepsilon) > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and $q = q_k(\varepsilon)$ there exists a radial singular fast decay solution u of (1.1). Moreover, there exist positive constants β_j, $j = 1 \dots k$ which depend on N and k, and the solution u that has the form

(1.5)

$$u(x) = K_q |x|^{-\frac{2}{q-1}} \left[\gamma_N \sum_{j=1}^{k} \left(\frac{1}{|x|^2 + (\beta_j \varepsilon^{(j-1)})^{\frac{4}{N-2}}} \right)^{\frac{N-2}{2} - \frac{1}{q-1}} \varepsilon^{(j-1)} \beta_j \, (1 + o(1)) \right],$$

and $q_k(\varepsilon)$ satisfies

$$\left(\frac{1}{q_k(\varepsilon) - 1} \right)^{N/2} = c_N k \varepsilon + o(\varepsilon) \qquad \text{as } \varepsilon \to 0$$

where $\gamma_N = (N(N-2))^{\frac{N-2}{4}}$, $c_N = \frac{1}{2}(\frac{N-2}{2})^{\frac{N+2}{2}} \frac{\Gamma(\frac{N}{2})}{\Gamma(N)}$, and $o(1) \to 0$ uniformly on $\mathbb{R}^N$ as $\delta \to 0$. Note here that $\beta_1 = \gamma_N^{-1}$.

2. Scheme of the proof of Theorem 1.1

Consider the Emden-Fowler change of variables:

$$v(t) = r^{\alpha} u(r), \quad r = e^t,$$

where

$$\alpha = \frac{N-2}{2}.$$

Then the equation $\Delta u + u^p + u^q = 0$ in $\mathbb{R}^N$ is equivalent to

$$v'' - \alpha^2 v + e^{\sigma_p t} v^p + e^{\sigma_q t} v^q = 0 \quad \text{in } \mathbb{R}, \tag{2.1}$$

where

$$\sigma_p = \alpha + 2 - \alpha p \quad \text{and} \quad \sigma_q = \alpha + 2 - \alpha q.$$

We note that equation (2.1) is the Euler-Lagrange equation of the functional

$$I(v) = \int_{-\infty}^{\infty} \left(\frac{1}{2}(v')^2 + \frac{\alpha^2}{2} v^2 - e^{\sigma_p t} \frac{|v|^{p+1}}{p+1} - e^{\sigma_q t} \frac{|v|^{q+1}}{q+1} \right) dt. \tag{2.2}$$

When p and q are given by (1.4) we have the expressions

$$\sigma_p = 1 - \alpha\varepsilon, \qquad \sigma_q = -\alpha\delta.$$

In the sequel we always work with

$$\frac{1}{C}\delta \leq \varepsilon \leq C\delta,$$

for some fixed $C > 0$.

For small $\varepsilon, \delta > 0$ a first approximation to a solution of (2.1) is given by

$$U_0(t) = \gamma_N 2^{-\alpha} \cosh(t)^{-\alpha}, \tag{2.3}$$

where

$$\gamma_N = (N(N-2))^{\frac{N-2}{4}}. \tag{2.4}$$

This function satisfies

$$U_0'' - \alpha^2 U_0 + U_0^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}.$$

In the original variables this function is the *standard bubble*

$$u_0(x) = \gamma_N \frac{1}{(1+|x|^2)^{(N-2)/2}},$$

which satisfies

$$\Delta u_0 + u_0^{\frac{N+2}{N-2}} = 0, \quad u_0 > 0 \quad \text{in } \mathbb{R}^N.$$

Thus U_0 corresponds to a function with fast decay. The translate $U_0(t-\xi)$ becomes a good approximation of (2.1) as $\xi \to -\infty$ and $\varepsilon, \delta \to 0$. To achieve an approximation with slow decay, we define $\beta = \frac{1}{p-1}$ and

$$U(t) = \gamma_N \frac{e^{\alpha t}}{(1+e^{2t})^{\beta}}, \quad t \in \mathbb{R}.$$

This is an actual solution of (2.1) if $q = 2p-1$, with one *bump*. As in [4] and [2], one can find a *multibump* solution starting from

$$V(t) = \sum_{j=1}^{k} U(t-\xi_j), \tag{2.5}$$

where $\xi_j \in \mathbb{R}$ are parameters to be adjusted. After a change of variables this is at main order the solution in the statement of Theorem 1.1.

The location of these points can be determined by an expansion of $I(V)$. Indeed, assuming the points are sufficiently separated we have

$$I(V) = -c_1 \sum_{j=1}^{k} e^{\xi_j} - c_2 \sum_{j=1}^{k-1} e^{\alpha(\xi_{j+1}-\xi_j)} + c_3 \delta \sum_{j=1}^{k} \xi_j + kc_0 + A\delta + o(\delta)$$

where c_1, c_2, c_3, A, c_0 are constants, see Proposition 3.1, where also the constants are given. Note that $c_1, c_2, c_3 > 0$. One can see that in order to obtain a solution, $\xi_1, \dots, \xi_k$ has to be close to a critical point of the above functional. To see this criticality more clearly it is convenient to write

$$\begin{cases} \xi_1 = \log\delta - \log\Lambda_1, \\ \xi_{j+1} = \xi_j + \dfrac{1}{\alpha}\log\delta - \log\Lambda_{j+1} \quad \forall j = 1, \dots, k-1, \end{cases} \tag{2.6}$$

with

$$\frac{1}{M} \le \Lambda_j \le M \quad \forall j = 1, \dots, k. \tag{2.7}$$

and $M > 1$ a constant to be fixed later on. Note that

$$\xi_j = \left(1 + \frac{j-1}{\alpha}\right)\log\delta - \sum_{i=1}^{j} \log\Lambda_i \quad \text{for } j = 1, \dots, k.$$

With this choice of the points, $I(V)$ takes the form

$$-c_1\delta\Lambda_1^{-1}-c_2\delta\sum_{j=2}^{k}\Lambda_j^{-\alpha}-c_3\delta\sum_{j=1}^{k}(k-j+1)\log\Lambda_j+c_3k(1+\frac{k-1}{\alpha})\delta\log\delta+kc_0+A\delta+o(\delta).$$

Let

$$\varphi(\Lambda_1,\ldots,\Lambda_k)=\frac{c_1}{\Lambda_1}+c_3k\log\Lambda_1+\sum_{j=2}^{k}\left(c_2\Lambda_j^{-\alpha}+(k-j+1)c_3\log\Lambda_j\right).$$

We note that φ has a unique critical point $\Lambda^*=(\Lambda_1^*,\ldots,\Lambda_k^*)$ given by

$$\Lambda_1^*=\frac{c_1}{kc_3},\qquad \Lambda_j^*=\left(\frac{c_2\alpha}{c_3(k-j+1)}\right)^{1/\alpha}\quad \forall j=2,\ldots,k, \tag{2.8}$$

and that this critical point is a nondegenerate minimum. In the sequel we fix the number M in (2.7) such that $\Lambda_i^*\in(\frac{1}{2M},2M)$.

To find an actual solution v close to V we perform a Lyapunov-Schmidt reduction. We look for a solution v to (2.1) of the form

$$v=V+\phi,$$

where ϕ is a lower order correction. We find the following equation for ϕ

$$L\phi+E+N(\phi)=0\quad\text{in }\mathbb{R}, \tag{2.9}$$

where

$$L\phi=\phi''-\alpha^2\phi+(pe^{\sigma_p t}V^{p-1}+qe^{\sigma_q t}V^{q-1})\phi \tag{2.10}$$

$$N(\phi)=e^{\sigma_p t}((V+\phi)^p-V^p-pV^{p-1}\phi)+e^{\sigma_q t}((V+\phi)^q-V^q-qV^{q-1}\phi)$$

$$E=V''-\alpha^2V+e^{\sigma_p t}V^p+e^{\sigma_q t}V^q. \tag{2.11}$$

The perturbation $\phi:\mathbb{R}\to\mathbb{R}$ will be small in an appropriate norm, which we introduce next. Let $\tau>0$ be a small fixed number, $0<\nu<\min(2,\alpha)$ and define

$$\|\phi\|_*=\sup_{t\in\mathbb{R}}\frac{|\phi(t)|}{w(t)},$$

where

$$w(t)=\begin{cases}e^{-(\alpha+\tau)(t-\xi_1)} & \text{if } t\geq\xi_1\\ \displaystyle\sum_{i=1}^{k}e^{-\nu|t-\xi_i|} & \text{if } t\leq\xi_1.\end{cases} \tag{2.12}$$

To motivate this choice of norm, we remark that the exponential decay of ϕ between the points ξ_j is expected because far from these points the dominant terms in the equation (2.1) are the linear ones, i.e. $\phi''-\alpha^2\phi$. Bounded solutions will then have exponential decay away from the ξ_j of the form $e^{-\nu|t-\xi_i|}$ with $0<\nu<\alpha$. For $t\geq\xi_1$ one in general can expect the same behavior. However,the solution we are looking for has slow decay as $t\to+\infty$, in the sense that it behaves as $e^{(\alpha-2\beta)t}$ as $t\to+\infty$ where $\beta=\frac{1}{p-1}$. To solve the nonlinear problem by the Banach fixed point theorem we need ϕ to have faster decay than $e^{-\alpha t}$ as $t\to+\infty$. To see this, consider the following term in $N(\phi)$:

$$e^{\sigma_p t}((V+\phi)^p-V^p-pV^{p-1}\phi)\sim e^{\sigma_p t}V^{p-2}\phi^2\sim e^{\sigma_p t+(\alpha-2\beta)(p-2)t}\phi^2$$

where $\beta = \frac{1}{p-1}$. Note that $\sigma_p + (\alpha - 2\beta)(p-2) = \beta - \alpha = \alpha + O(\varepsilon)$. Suppose that ϕ has decay of the form $|\phi(t)| \le Ae^{-m|t-\xi_1|}$ for $t \ge \xi_1$. Then in $N(\phi)$ we find one term of the form

$$CA^2 e^{(\alpha+O(\varepsilon))\xi_1}\, e^{(\alpha+O(\varepsilon)-2m)(t-\xi_1)} \quad t \ge \xi_1$$

For the contraction mapping principle to work we would like m such that $\alpha + O(\varepsilon) - 2m \le -m$ for all $\varepsilon > 0$ small, which leads to the choice $m = \alpha + \tau$, $\tau > 0$ fixed.

Having introduced a good norm for the contraction mapping to work, let us look at the error E defined by (2.11). We note that E contains a term of the form $Se^{(\alpha-2\beta)t}$, which we may call the slowly decaying part of E, and other terms that decay faster, where S is a function of $\varepsilon, \delta, \Lambda_1, \ldots, \Lambda_k$. Since $\beta = \alpha + O(\varepsilon)$, we see that $\|E\|_* = +\infty$ unless $S = 0$. We prove in Proposition 4.1 that there is a function $\varepsilon_k(\delta, \Lambda_1, \ldots, \Lambda_k) > 0$ such that $S = 0$ if $\varepsilon = \varepsilon_k(\delta, \Lambda_1, \ldots, \Lambda_k)$, and then also $\|E\|_* \le C\delta^\theta$ for some $\theta > 1/2$ and all $\delta > 0$ small. The function $\varepsilon_k(\delta, \Lambda_1, \ldots, \Lambda_k)$ is at main order of the form $C\delta/\Lambda_1$ for some constant C.

Using the contraction mapping principle, and a suitable right inverse of L, constructed in Section 5, which preserves the norm $\|\ \|_*$, we prove in Section 6 that for $\delta > 0$ small enough and $\varepsilon = \varepsilon_k(\Lambda, \delta)$ there exists a solution ϕ to the nonlinear projected problem

$$L\phi + E + N(\phi) = \sum_{i=1}^{k} c_i \tilde{Z}_i,$$

such that $\|\phi\|_* \le A\delta^\theta$, for a suitable constant $A > 0$. Here $\tilde{Z}_i$ are defined in (5.2). Finally, to find a solution of (2.9) it remains to verify that one can choose $\Lambda = (\Lambda_1, \ldots, \Lambda_k)$ such that the constants c_i are all zero, which is done in Section 6.

3. Expansion of the energy

Proposition 3.1. *Let $M > 1$. Assume $\Lambda = (\Lambda_1, \ldots, \Lambda_k)$ satisfies $\Lambda_i \in [1/M, M]$, $i = 1, \ldots, k$ and $\xi_1, \ldots, \xi_k$ are given by (2.6) . Let V denote the initial ansatz (2.5) and I the functional (2.2). Assuming $0 < \varepsilon \le C\delta$ for some C, we have*

$$I(V) = -\delta\varphi(\Lambda) + kc_0 + A\delta + B\delta\log\delta + \delta\Theta_0(\Lambda)$$

where

$$\varphi(\Lambda_1, \ldots, \Lambda_k) = \frac{c_1}{\Lambda_1} + c_3 k \log\Lambda_1 + \sum_{j=2}^{k}\left(c_2\Lambda_j^{-\alpha} + (k-j+1)c_3\log\Lambda_j\right),$$

and $\Theta_0 \to 0$ in C^1 norm on the set defined by $\Lambda_i \in [1/M, M]$, $i = 1, \ldots, k$.

The constants are given by

$$c_1 = \frac{N-2}{2N-2}\int_{-\infty}^{\infty} e^t U_0^{\frac{2N-2}{N-2}}\,dt \qquad c_2 = \frac{\gamma_N}{2}\int_{-\infty}^{\infty} U_0(t)^{2^*-1}e^{\alpha t}\,dt \tag{3.1}$$

$$c_3 = \frac{\alpha}{2^*}\int_{-\infty}^{\infty} U_0^{2^*}\,dt, \tag{3.2}$$

$$c_0 = \frac{1}{2}\int_{-\infty}^{\infty}((U_0')^2+\alpha^2 U_0^2) - \frac{1}{2^*}\int_{-\infty}^{\infty} U_0^{2^*},$$

$$A = \frac{k}{(2^*)^2}\int_{-\infty}^{\infty} U_0^{2^*} - \frac{k}{2^*}\int_{-\infty}^{\infty} U_0^{2^*}\log U_0, \qquad B = \frac{\alpha}{2^*}k\left(1+\frac{k-1}{2\alpha}\right)\int_{-\infty}^{\infty} U_0^{2^*},$$

Here $2^* = 2N/(N-2)$. Note that $c_1, c_2, c_3 > 0$ and these constants can be explicitly computed using the identity

$$\int_{-\infty}^{\infty}\cosh(s)^{-q}\, e^{-\mu\, s}\, ds = 2^{q-1}\frac{\Gamma(\frac{q-\mu}{2})\,\Gamma(\frac{q+\mu}{2})}{\Gamma(q)}$$

for all $\mu\in\mathbb{R}$ and $q > \max\{\mu, -\mu\}$.

Proof of Proposition 3.1. We follow the computation in [4]. We write

$$I = I_1 + I_2 + I_3 + I_4 + I_5$$

where

$$I_1(v) = \int_{-\infty}^{\infty}\left(\frac{1}{2}(v')^2 + \frac{\alpha^2}{2}v^2 - \frac{|v|^{2^*}}{2^*}\right)$$

$$I_2(v) = \frac{1}{2^*}\int_{-\infty}^{\infty}(|v|^{2^*} - |v|^{q+1}) + \left(\frac{1}{2^*}-\frac{1}{q+1}\right)\int_{-\infty}^{\infty} e^{\sigma_q t}|v|^{q+1}$$

$$I_3(v) = \int_{-\infty}^{\infty}\left(1-e^{\sigma_q t}\right)\frac{|v|^{q+1}}{2^*} \qquad\qquad I_4(v) = -\int_{-\infty}^{\infty} e^{\sigma_p t}\frac{|v|^{p+1}}{p+1}.$$

Let us start with the computation of $I_2(V)$. Since $q = \frac{N+2}{N-2}+\delta$, we have

$$\frac{1}{2^*}\int_{-\infty}^{\infty}(V^{2^*}-V^{q+1}) = -\frac{\delta}{2^*}\int_{-\infty}^{\infty}V^{2^*}\log V + o(\delta) = -\frac{k\delta}{2^*}\int_{-\infty}^{\infty}U^{2^*}\log U + o(\delta)$$
$$= -\frac{k\delta}{2^*}\int_{-\infty}^{\infty}U_0^{2^*}\log U_0 + o(\delta),$$

recalling that U depends on ε and $\varepsilon = O(\delta)$. The second term in $I_2(V)$ is

$$\frac{\delta}{(2^*)^2}\int_{-\infty}^{\infty} e^{\sigma_q t}V^{q+1} + o(\delta) = \frac{\delta}{(2^*)^2}\int_{-\infty}^{\infty} e^{\sigma_q t}V^{2^*} + o(\delta) = \frac{\delta k}{(2^*)^2}\int_{-\infty}^{\infty} U_0^{2^*} + o(\delta).$$

Therefore

$$I_2(V) = A\delta + o(\delta). \tag{3.3}$$

Regarding $I_3(V)$ we have

$$I_3(V) = \frac{\alpha\delta}{2^*}\int_{-\infty}^{\infty} tV(t)^{q+1}\,dt + o(\delta) = \frac{\alpha\delta}{2^*}\sum_{i=1}^{k}\int_{-\infty}^{\infty} tU(t-\xi_i)^{q+1}\,dt + o(\delta)$$
$$= \frac{\alpha\delta}{2^*}\sum_{i=1}^{k}\xi_i\int_{\infty}^{\infty} U_0^{2^*} + o(\delta).$$

Since ξ_i are given by (2.6) we obtain

$$I_3(V) = \frac{\alpha\delta}{2^*}\int_{\infty}^{\infty} U_0^{2^*}\left[k\left(1+\frac{k-1}{2\alpha}\right)\log\delta - \sum_{i=1}^{k}(k-i+1)\log\Lambda_i\right] + o(\delta). \tag{3.4}$$

For $I_4(V)$ we see that

$$\begin{aligned} I_4(V) &= -\frac{1}{p+1}\int_{-\infty}^{\infty} e^{\sigma_p t} V^{p+1} = -\frac{1}{p+1}\int_{-\infty}^{\infty} e^{\sigma_p t} U(t-\xi_1)^{p+1} + o(\delta) \\ &= -\frac{e^{\sigma_p \xi_1}}{p+1}\int_{-\infty}^{\infty} e^{\sigma_p t} U(t)^{p+1} + o(\delta) = -\frac{\delta\Lambda_1^{-1}}{p+1}\int_{-\infty}^{\infty} e^{\sigma_p t} U(t)^{p+1} + o(\delta) \\ &= -\frac{\delta}{\Lambda_1}\frac{N-2}{2N-2}\int_{-\infty}^{\infty} e^t U_0(t)^{\frac{2N-2}{N-2}}\, dt + o(\delta). \end{aligned} \tag{3.5}$$

Finally we compute $I_1(V)$. Using the notation $U_i(t) = U(t-\xi_i)$ we have

$$\begin{aligned} I_1(V) &= \int_{-\infty}^{\infty} \left(\frac{1}{2}(\sum_i U_i')^2 + \frac{\alpha^2}{2}(\sum_i U_i)^2 - \frac{1}{2^*}(\sum_i U_i)^{2^*} \right) \\ &= kI_U + \frac{1}{2}\sum_{i\neq j}\int_{-\infty}^{\infty}(-U_i'' + \alpha^2 U_i)U_j - \frac{1}{2^*}\int_{-\infty}^{\infty}\left[(\sum_i U_i)^{2^*} - \sum_i U_i^{2^*} \right], \end{aligned}$$

where we have set

$$I_U = \frac{1}{2}\int_{-\infty}^{\infty}((U')^2 + \alpha^2 U^2) - \frac{1}{2^*}\int_{-\infty}^{\infty} U^{2^*}.$$

Note that

$$I_U = c_0 + o(\delta) \quad \text{as } \delta \to 0. \tag{3.6}$$

Indeed, I_U is a function of ε and

$$\frac{d}{d\varepsilon} I_U = \int_{-\infty}^{\infty} (-U'' + \alpha^2 U - U^{2^*-1})\frac{\partial U}{\partial \varepsilon}$$

so that $\frac{d}{d\varepsilon} I_U = 0$ at $\varepsilon = 0$. Let $F_i(t) = F(t-\xi_i)$ where $F = -U'' + \alpha^2 U - U^{2^*-1}$. Then by (3.6)

$$I_1(V) = \frac{1}{2}\sum_{i\neq j}\int_{-\infty}^{\infty}(U_i^{2^*-1} + F_i)U_j - \frac{1}{2^*}\int_{-\infty}^{\infty}\left[(\sum_i U_i)^{2^*} - \sum_i U_i^{2^*} \right] + kc_0 + o(\delta).$$

Let

$$t_1 = 0, \quad t_j = (1 + \frac{j-1/2}{\alpha})\log\delta, \quad j = 2,\ldots,k-1, \qquad t_k = -\infty.$$

Then we can write

$$I_1(V) = -\frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}\int_{t_i}^{t_{i-1}} U_i^{2^*-1}U_j + kc_0 + R$$

where R contains all the rest and is given by

$$\begin{aligned} R = \frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}\int_{t_i}^{t_{i-1}} F_i U_j + \frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}\int_{\mathbb{R}\setminus[t_i,t_{i-1}]} U_i^{2^*-1}U_j + \frac{1}{2^*}\sum_{i=1}^{k}\sum_{j\neq i}\int_{t_i}^{t_{i-1}} U_j^{2^*} \\ - \frac{1}{2^*}\sum_{i=1}^{k}\int_{t_i}^{t_{i-1}}\left[(U_i + \sum_{j\neq i} U_j)^{2^*} - U_i^{2^*} - 2^* U_i^{2^*-1}\sum_{j\neq i} U_j \right] + o(\delta). \end{aligned}$$

We note that if $|i-j| \geq 2$ then

$$\int_{t_i}^{t_{i-1}} U_i^{2^*-1} U_j = o(\delta).$$

We have the expansions

$$U(t) = \gamma_N e^{\alpha t}(1 + O(e^{2t})) \quad \text{as } t \to -\infty,$$

$$U(t) = \gamma_N e^{(\alpha-2\beta)t}(1 + O(e^{-2t})) \quad \text{as } t \to +\infty.$$

Therefore, for $i = 1, \ldots, k-1$ and $j = i+1$

$$\begin{aligned}
\int_{t_i}^{t_{i-1}} U_i^{2^*-1} U_j &= \int_{t_i-\xi_i}^{t_{i-1}-\xi_i} U(t)^{\frac{N+2}{N-2}} U(t - (\xi_{i+1} - \xi_i))\, dt \\
&= \gamma_N \int_{t_i-\xi_i}^{t_{i-1}-\xi_i} U(t)^{\frac{N+2}{N-2}} e^{(\alpha-2\beta)(t-(\xi_{i+1}-\xi_i))}(1 + O(e^{2(t-(\xi_{i+1}-\xi_i))}))\, dt \\
&= \gamma_N \delta^{\frac{2\beta-\alpha}{\alpha}} \Lambda_{i+1}^{\alpha-2\beta} \int_{t_i-\xi_i}^{t_{i-1}-\xi_i} U(t)^{\frac{N+2}{N-2}} e^{(\alpha-2\beta)t}(1 + O(e^{2(t-(\xi_{i+1}-\xi_i))}))\, dt \\
&= \gamma_N \delta \Lambda_{i+1}^{-\alpha} \int_{-\infty}^{\infty} U_0(t)^{\frac{N+2}{N-2}} e^{-\alpha t}\, dt + o(\delta).
\end{aligned}$$

A similar calculation shows that if $i = 2, \ldots, k$ and $j = i-1$ then

$$\int_{t_i}^{t_{i-1}} U_i^{2^*-1} U_j = \gamma_N \delta \Lambda_i^{-\alpha} \int_{-\infty}^{\infty} U_0(t)^{\frac{N+2}{N-2}} e^{\alpha t}\, dt + o(\delta).$$

Regarding R, it is possible to verify that

$$R = o(\delta)$$

as $\delta \to 0$. For instance we have:

$$\begin{aligned}
\int_{t_i}^{t_{i-1}} \Big[(U_i + \sum_{j\neq i} U_j)^{2^*} - U_i^{2^*} - 2^* U_i^{2^*-1} \sum_{j\neq i} U_j\Big] &\leq C \int_{t_i}^{t_{i-1}} U_i^{2^*-2} \sum_{j\neq i} U_j^2 \\
&\leq C \int_0^{\frac{1}{2\alpha}|\log\delta|} e^{-\alpha\frac{4}{N-2}t} e^{-2\alpha(\frac{1}{\alpha}|\log\delta| - t)}\, dt \\
&\leq C\delta^2 \int_0^{\frac{1}{2\alpha}|\log\delta|} e^{2\alpha\frac{N-4}{N-2}t}\, dt
\end{aligned}$$

which is $O(\delta^{1+\frac{2}{N-2}})$ if $N \geq 5$, $O(\delta^2|\log\delta|)$ if $N = 4$, and $O(\delta^2)$ if $N = 3$. The other terms in R can be handled similarly. Therefore, we obtain

$$I_1(V) = -\delta\gamma_N \int_{-\infty}^{\infty} U_0(t)^{\frac{N+2}{N-2}} e^{\alpha t}\, dt \sum_{j=2}^{k} \Lambda_j^{-\alpha} + kc_0 + o(\delta). \tag{3.7}$$

Combining (3.3), (3.4), (3.5), and (3.7) we arrive at

$$I(V) = -\delta\varphi(\Lambda) + kc_0 + A\delta + B\delta\log\delta + o(\delta)$$

as $\delta \to 0$ for some constants A, B, with $o(\delta)$ uniformly in the region $\Lambda_i \in [1/M, M]$, $i = 1, \ldots, k$. A similar calculation shows that this expansion is also valid in the C^1 norm with respect to $\Lambda = (\Lambda_1, \ldots, \Lambda_k)$. □

4. Error estimate

Here we will prove the following result.

Proposition 4.1. *Let $k \ge 2$ be an integer and fix $M > 1$. Suppose $\Lambda_i \in [1/M, M]$, $i = 1, \dots, k$, ξ_i are given by (2.6) and E is defined by (2.11). If $\nu > 0$ and $\tau > 0$ are chosen small in (2.12), there exists $\delta_0 > 0$, $\theta > 1/2$ and a function $\varepsilon_k(\delta, \Lambda_1, \dots, \Lambda_k) > 0$ such that for $0 < \delta \le \delta_0$ and $\varepsilon = \varepsilon_k(\delta, \Lambda_1, \dots, \Lambda_k)$*

$$\|E\|_* \le C\delta^\theta,$$

with C independent of δ.

The function ε_k is C^1 and satisfies

$$\varepsilon_k(\delta, \Lambda_1, \dots, \Lambda_k) = \frac{\gamma_N^{p-1}}{4\alpha^3\Lambda_1}\delta + o(\delta) \quad \text{as } \delta \to 0, \tag{4.1}$$

$$\frac{\partial \varepsilon_k}{\partial \Lambda_i}(\delta, \Lambda_1, \dots, \Lambda_k) = O(\delta) \quad \text{as } \delta \to 0, \tag{4.2}$$

where $o(\delta)$, $O(\delta)$ are uniform in the region $\Lambda_i \in [1/M, M]$, $i = 1, \dots, k$.
Proof. Let us write

$$U_j(t) = U(t - \xi_j) \quad \text{and} \quad V = \sum_{j=1}^k U_j.$$

We decompose

$$E = \sum_{j=1}^k E_j + A + B,$$

where

$$E_j = U_j'' - \alpha^2 U_j + e^{\sigma_p t} U_j^p + e^{\sigma_q t} U_j^q$$

$$A = e^{\sigma_p t}\Big(V^p - \sum_{j=1}^k U_j^p\Big) \quad , \quad B = e^{\sigma_q t}\Big(V^q - \sum_{j=1}^k U_j^q\Big).$$

Let

$$\beta = \frac{1}{p-1} = \alpha - \varepsilon\alpha^2 + O(\varepsilon^2).$$

Then a computation shows that

$$\begin{aligned} U_j'' - \alpha^2 U_j + e^{\sigma_p t} U_j^p = \left[e^{\sigma_p \xi_j}\gamma_N^p + 4\gamma_N\beta(\beta - \alpha)\right] \frac{e^{(\alpha+2)(t-\xi_j)}}{(1 + e^{2(t-\xi_j)})^{\beta+1}} + \\ - 4\gamma_N\beta(\beta+1)\frac{e^{(\alpha+2)(t-\xi_j)}}{(1 + e^{2(t-\xi_j)})^{\beta+2}}. \end{aligned}$$

Note that the terms $U_j'' - \alpha^2 U_j + e^{\sigma_p t} U_j^p$ of E_j have slow decay, i.e., $U_j'' - \alpha^2 U_j + e^{\sigma_p t} U_j^p \sim e^{(\alpha - 2\beta)t}$ as $t \to \infty$. We define the slowly decaying part of E_j as

$$S_j = \chi_{[t \ge \xi_1/2]}\left(4\gamma_N\beta(\beta - \alpha)e^{-(\alpha-2\beta)\xi_j} + \gamma_N^p e^{-(\alpha-2\beta)p\xi_j}\right)e^{(\alpha-2\beta)t},$$

where $\chi_{[t\ge\xi_1/2]}$ is the indicator of the set $[\xi_1/2, +\infty)$, and also

$$\tilde{E}_j = E_j - S_j.$$

The term A has also a slowly decaying part S_A given by

$$S_A = \chi_{[t\geq \xi_1/2]}\gamma_N^p \left[\Big(\sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j} \Big)^p - \sum_{j=1}^k e^{-(\alpha-2\beta)p\xi_j} \right] e^{(\alpha-2\beta)t}.$$

We define

$$\tilde{A} = A - S_A.$$

Given $\delta > 0$ small and $\xi_1, \ldots, \xi_k$ satisfying (2.6) and (2.7) we choose $\varepsilon > 0$ such that

$$\sum_{j=1}^k S_j + S_A = 0$$

which is equivalent to

$$0 = 4\gamma_N\beta(\beta-\alpha)\sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j} + \gamma_N^p \Big(\sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j} \Big)^p. \tag{4.3}$$

By (2.6) we see that at main order (in ε and δ) this equation has the form

$$4\gamma_N\beta(\beta-\alpha)e^{-(\alpha-2\beta)\xi_1} + \gamma_N^p e^{-(\alpha-2\beta)p\xi_1} = 0,$$

so that we have the asymptotic expansion (4.1). The estimate (4.2) follows also from (4.3).

We claim that

$$\|\tilde{E}_j\|_* \leq C\delta^{1-2\tau} \tag{4.4}$$

for $\delta > 0$ small. Consider separately the regions $t \geq \xi_1/2$, and $t \leq \xi_1/2$. Using the formula of S_j we have

$$\tilde{E}_j = O(e^{(\alpha-2\beta-2)(t-\xi_j)}) + O(e^{\sigma_p t} e^{((\alpha-2\beta)p-2)(t-\xi_j)}) + O(e^{\sigma_q t} e^{(\alpha-2\beta)q(t-\xi_j)})$$

for $t \geq \xi_1/2$ and we see from here that

$$\sup_{t\geq \xi_1/2} |\tilde{E}_j| e^{(\alpha+\tau)(t-\xi_1)} \leq C\delta^{1-\tau/2+O(\delta)} \leq C\delta^{1-\tau}$$

for $\delta > 0$ small.

We estimate now in the interval $t \leq \xi_1/2$. In this interval

$$\begin{aligned} \tilde{E}_j &= -4\gamma_N\beta(\alpha+1)\frac{e^{(\alpha+2)(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta+1}} + 4\gamma_N\beta(\beta+1)\frac{e^{(\alpha+4)(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta+2}} \\ &\quad + e^{\sigma_p t}\gamma_N^p \frac{e^{\alpha p(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta p}} + e^{\sigma_q t}\gamma_N^q \frac{e^{\alpha q(t-\xi_j)}}{(1+e^{2(t-\xi_j)})^{\beta q}} \\ &= O(\delta)(1+|t|+|\xi_j|)e^{-(\alpha+O(\delta))|t-\xi_j|}, \end{aligned} \tag{4.5}$$

where in this formula $O(\delta)$ designates a quantity bounded by a constant times δ. From (4.5) we have

$$\sup_{t\leq \xi_1} e^{\nu|t-\xi_i|}|\tilde{E}_j| \leq C\delta \quad \text{for any } i = 1, \ldots, k, \tag{4.6}$$

and

$$\sup_{\xi_1 \leq t \leq \xi_1/2} e^{(\alpha+\tau)|t-\xi_1|}|\tilde{E}_j| \leq C\delta^{1-2\tau} \tag{4.7}$$

for $\delta >$ small. Using (4.6), (4.7) we deduce (4.4).

Similarly we estimate $\tilde A$ first in the region $t \ge \xi_1/2$. For $t \ge \xi_1/2$

$$\tilde A = A - S_A = A_1 + A_2,$$

where

$$A_1 = \gamma_N^p e^{\sigma_p t} e^{(\alpha-2\beta)pt}\left[\left(\sum_{j=1}^k \frac{e^{-(\alpha-2\beta)\xi_j}}{(1+e^{2(\xi_j-t)})^\beta}\right)^p - \left(\sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j}\right)^p\right]$$

and

$$A_2 = -\gamma_N^p e^{\sigma_p t} e^{(\alpha-2\beta)pt}\left[\sum_{j=1}^k \frac{e^{-(\alpha-2\beta)p\xi_j}}{(1+e^{2(\xi_j-t)})^{\beta p}} - \sum_{j=1}^k e^{-(\alpha-2\beta)p\xi_j}\right].$$

In the range $t \ge \xi_1/2$ we have $(1+se^{2(\xi_j-t)})^{-\beta-1} = O(1)$ so by the mean value theorem

$$|A_1| \le Ce^{\sigma_p t} e^{(\alpha-2\beta)pt}\sum_{j=1}^k e^{-(\alpha-2\beta)\xi_j p} e^{2(\xi_j - t)} \quad \text{for } t \ge \xi_1/2.$$

We then compute

$$\sup_{t\ge \xi_1/2} e^{(\alpha+\tau)(t-\xi_1)}|A_1| \le C\delta^{2-\tau/2+O(\delta)}.$$

Similarly

$$\sup_{t\ge \xi_1/2} e^{(\alpha+\tau)(t-\xi_1)}|A_2| \le C\delta^{2-\tau/2+O(\delta)}$$

and we deduce

$$\sup_{t\ge \xi_1/2} e^{(\alpha+\tau)(t-\xi_1)}|\tilde A| \le C\delta^{2-\tau/2+O(\delta)}. \tag{4.8}$$

In the region $\xi_1 \le t \le \xi_1/2$ we have

$$|A| \le Ce^{\sigma_p t}\left[\sum_{j=2}^k U(t-\xi_j)U(t-\xi_1)^{p-1} + \sum_{j=2}^k U(t-\xi_j)^p\right]$$

which gives

$$\sup_{\xi_1\le t\le \xi_1/2} e^{(\alpha+\tau)(t-\xi_1)}|A| \le C\delta^{2-\tau/2+O(\delta)}. \tag{4.9}$$

To estimate the term A for $t \le \xi_1$, using that $e^{\sigma_p t} \le C\delta^{1+O(\delta)}$ in this region we see that

$$\sup_{t\le \xi_1} e^{\nu|t-\xi_i|}|A| \le C\delta^{1+O(\delta)} \quad \text{for } i = 1,\dots,k. \tag{4.10}$$

Hence by (4.8), (4.9) and (4.10) we find

$$\|\tilde A\|_* \le C\delta^{1+O(\delta)}.$$

We finally estimate $\|B\|_*$. We claim that there is $\theta > 1/2$ such that

$$\|B\|_* \le C\delta^\theta \tag{4.11}$$

for $\delta > 0$ sufficiently small. Indeed, let $i = 1,\dots,k-1$ and let us estimate

$$\sup_{\xi_{i+1}\le t\le \xi_i} (e^{\nu|t-\xi_{i+1}|} + e^{\nu|t-\xi_i|})|B|.$$

Let $\lambda \in (0, 1/2)$ to be fixed later on. We consider the 3 intervals

$$\begin{aligned} I_1 &= [\xi_{i+1}, (1-\lambda)\xi_{i+1} + \lambda\xi_i], \\ I_2 &= [(1-\lambda)\xi_{i+1} + \lambda\xi_i, \lambda\xi_{i+1} + (1-\lambda)\xi_i], \\ I_3 &= [\lambda\xi_{i+1} + (1-\lambda)\xi_i, \xi_i]. \end{aligned}$$

The worst term in each sum of B is $U(t-\xi_{i+1})^q$ or $U(t-\xi_i)^q$. We estimate

$$\begin{aligned} \sup_{t\in I_2} e^{\nu|t-\xi_i|} U(t-\xi_i)^q &\le C \sup_{t\in I_2} e^{\nu(\xi_i - t)} e^{-\alpha q(\xi_i - t)} \\ &\le C e^{(\nu - \alpha q)\xi_i} \sup_{t\in I_2} e^{(\alpha q - \nu)t} \\ &= C\delta^{\lambda(q-\nu/\alpha)}. \end{aligned}$$

Since $q > 1$ we may choose $\nu > 0$ small so that $q - \nu/\alpha > 1$. Then take $\lambda \in (0, 1/2)$ so that

$$\lambda(q - \nu/\alpha) > \frac{1}{2}. \tag{4.12}$$

We also have

$$\sup_{t\in I_2} e^{\nu|t-\xi_{i+1}|} U(t-\xi_{i+1})^q \le C\delta^{\lambda(q-\nu/\alpha)}.$$

This gives

$$\sup_{t\in I_2} \frac{|B|}{w(t)} \le C\delta^{\lambda(q-\nu/\alpha)+O(\delta)}.$$

We now compute

$$\sup_{t\in I_3} e^{\nu|t-\xi_i|} e^{\sigma_q t} \left[\left(\sum_{j=1}^k U(t-\xi_j) \right)^q - \sum_{j=1}^k U(t-\xi_j)^q \right].$$

In this region $U(t-\xi_i)$ is dominant. So

$$\begin{aligned} \left(\sum_{j=1}^k U(t-\xi_j) \right)^q &= U(t-\xi_i)^q \left(1 + \sum_{j\ne i}^k \frac{U(t-\xi_j)}{U(t-\xi_i)} \right)^q \\ &= U(t-\xi_i)^q + \sum_{j\ne i}^k O(U(t-\xi_j)U(t-\xi_i)^{q-1}). \end{aligned}$$

So

$$\begin{aligned} &\sup_{t\in I_3} e^{\nu|t-\xi_i|} e^{\sigma_q t} \left| \left(\sum_{j=1}^k U(t-\xi_j) \right)^q - \sum_{j=1}^k U(t-\xi_j)^q \right| \\ &\le C \sup_{t\in I_3} e^{\nu|t-\xi_i|} e^{\sigma_q t} \left[\sum_{j\ne i}^k U(t-\xi_j) U(t-\xi_i)^{q-1} + \sum_{j=1}^k U(t-\xi_j)^q \right] \end{aligned}$$

The worst case is $j = i+1$ in the first sum

$$\begin{aligned} &\sup_{t\in I_3} e^{\nu|t-\xi_i|} e^{\sigma_q t} U(t-\xi_{i+1}) U(t-\xi_i)^{q-1} \\ &\le C e^{\nu\xi_i} e^{-(\alpha-2\beta)\xi_{i+1}} e^{-\alpha\xi_i(q-1)} \sup_{t\in I_3} e^{-\nu t} e^{\sigma_q t} e^{(\alpha-2\beta)t} e^{\alpha(q-1)t} \end{aligned}$$

If the sup is attained at $t = \xi_i$:

$$\sup_{t\in I_3} e^{\nu|t-\xi_i|} e^{\sigma_q t} U(t-\xi_{i+1}) U(t-\xi_i)^{q-1}$$
$$= C e^{\alpha(\xi_{i+1}-\xi_i)+O(\delta|\log\delta|)} \le C\delta$$

If the sup is attained at $t = \lambda\xi_{i+1} + (1-\lambda)\xi_i$:

$$\sup_{t\in I_3} e^{\nu|t-\xi_i|} e^{\sigma_q t} U(t-\xi_{i+1}) U(t-\xi_i)^{q-1}$$
$$\le C\delta^{(q-1-\nu/\alpha)\lambda}\delta^{(2\beta-\alpha)/\alpha(1-\lambda)} e^{\delta|\log\delta|}$$
$$\le C\delta^{(q-\nu/\alpha)\lambda+1-2\lambda}$$

Since $\lambda \in (0,1/2)$, we have that $(q-\nu/\alpha)\lambda + 1 - 2\lambda > 1/2$ by (4.12).

By similar estimates in the remaining intervals we obtain the validity of (4.11) with $\theta = \lambda(q-\nu/\alpha) > 1/2$. □

5. The linearized equation

In this section, given $\xi_1, \ldots, \xi_k \in \mathbb{R}$ such that (2.6) and (2.7) hold for some fixed $M > 1$, we study the linear problem

$$\begin{cases} L(\phi) = h + \displaystyle\sum_{i=1}^k c_i \tilde Z_i \quad \text{in } \mathbb{R}, \\ \displaystyle\lim_{t\to\pm\infty} \phi(t) = 0, \end{cases} \tag{5.1}$$

where L is the operator defined in (2.10), and $\tilde Z_i$ is defined by

$$\tilde Z_i(t) = U_0'(t-\xi_i)\eta(t-\xi_i), \tag{5.2}$$

where $\eta \in C^\infty(\mathbb{R})$ is an even cut-off function, $\eta \ge 0$, such that *supp*$(\eta) = [-R, R]$ where $R > 0$ is a fixed constant. We will also use the notation

$$Z_i(t) = U_0'(t-\xi_i).$$

The main result in this section is the following.

Proposition 5.1. *Fix $M > 1$ and assume $\xi_1, \ldots, \xi_k \in \mathbb{R}$ satisfy (2.6) and (2.7). Then there are $\delta_0 > 0$, C such that for $0 < \delta \le \delta_0$ and $0 < \varepsilon \le C\delta$ there is a linear operator T such that given h with $\|h\|_* < \infty$, $(\phi, c_1, \ldots, c_k) = T(h)$ solves (5.1). Moreover*

$$\|\phi\|_* \le C\|h\|_* \quad \textit{and} \quad |c_i| \le C\|h\|_* \quad \forall i = 1, \ldots, k.$$

For $\tau > 0$, $0 < \nu < \alpha$ fixed and $\phi : \mathbb{R} \to \mathbb{R}$ we define

$$\|\phi\|_1 = \sup_{t\ge\xi_1}(e^{(\alpha+\tau)(t-\xi_1)}|\phi(t)|) + \sup_{t\le\xi_1}(e^{\nu(\xi_1-t)}|\phi(t)|). \tag{5.3}$$

Lemma 5.2. *Assume $\tau > 0$ and $0 < \nu < \alpha$. Given h with $\|h\|_1 < +\infty$, there is a unique ϕ with $\|\phi\|_1 < +\infty$ and $c_1 \in \mathbb{R}$ such that*

$$\phi'' - \alpha^2\phi + 2^* U_0(t-\xi_1)^{2^*-1}\phi = h + c_1 \tilde Z_1 \quad \textit{in } \mathbb{R}. \tag{5.4}$$

Moreover, there is $C > 0$ such that

$$\|\phi\|_1 \le C\|h\|_1 \quad \textit{and} \quad |c_1| \le C\|h\|_1. \tag{5.5}$$

Proof. By translation we may assume here that $\xi_1 = 0$.

Let U_0 be the function defined by (2.3) and $z_1 = U_0'$. Then

$$z_1(t) = -\gamma_N 2^{-\alpha}\alpha \cosh(t)^{-N/2}\sinh(t) \tag{5.6}$$

and satisfies

$$z'' - \alpha^2 z + 2^* U_0^{2^*-1} z = 0 \quad \text{in } \mathbb{R}. \tag{5.7}$$

and

$$z_1(0) = 0 \quad \text{and} \quad z_1'(0) = -2^{-\alpha}\alpha\gamma_N.$$

Let z_2 be the solution to (5.7) with initial conditions

$$z_2(0) = 1 \quad \text{and} \quad z_2'(0) = 0.$$

To prove uniqueness observe that if $h = 0$, then multiplying (5.4) by z_1 we deduce that $c_1 = 0$. Then ϕ must be a linear combination of z_1 and z_2, and since $\|\phi\|_1 < +\infty$, $\phi = cz_1$ for some c. But again, because $\|\phi\|_1 < +\infty$, $\phi = 0$.

Let us prove the existence. Suppose $\|h\|_1 < \infty$ and $\int_{-\infty}^{\infty} hz_1 = 0$. Define

$$\phi(t) = \frac{2^\alpha}{\alpha\gamma_N}\left(z_1(t)\int_t^\infty z_2(s)h(s)\,ds - z_2(t)\int_t^\infty z_1(s)h(s)\,ds\right). \tag{5.8}$$

Then ϕ is a solution to the linear problem

$$\phi'' - \alpha^2\phi + 2^* U_0^{2^*-1}\phi = h \quad \text{in } \mathbb{R}$$

and

$$\|\phi\|_1 \le C\|h\|_1. \tag{5.9}$$

Indeed, from (5.6) we have $z_1(t) = ce^{-\alpha|t|} + o(e^{-\alpha|t|})$ as $t \to \pm\infty$ for some constant c. Furthermore one can also prove that $z_2(t) = c'e^{\alpha|t|} + o(e^{\alpha|t|})$ as $t \to \pm\infty$ for some constant $c' \neq 0$. Then (5.9) follows from (5.8) and the behaviors of z_1, z_2 at $\pm\infty$.

In the general case, when h is not necessarily orthogonal to z_1, we define

$$c_1 = -\frac{\int_{-\infty}^{\infty} hz_1}{\int_{-\infty}^{\infty} \tilde{Z}_1 z_1}$$

and apply the previous construction to $h + c_1\tilde{Z}_1$. □

For $\phi : \mathbb{R} \to \mathbb{R}$ consider the norm

$$\|\phi\|_2 = \sup_{t\in\mathbb{R}} \left(\sum_{i=2}^k e^{-\nu|t-\xi_i|}\right)^{-1} |\phi(t)|. \tag{5.10}$$

Lemma 5.3. *Suppose that in the definition of $\|\ \|_2$ we take $0 < \nu < \alpha$. Then there exist $\delta_0 > 0$, C such that if $0 < \delta \le \delta_0$ and $\|h\|_2 < \infty$ there is a unique solution ϕ with $\|\phi\|_2 < +\infty$ and $c_2, \dots, c_k \in \mathbb{R}$ of*

$$\begin{cases} \phi'' - \alpha^2\phi + 2^* \sum_{i=2}^k U_0(t-\xi_i)^{2^*-1}\phi = h + \sum_{i=2}^k c_i\tilde{Z}_i & \text{in } \mathbb{R} \\ \int_{\mathbb{R}} \phi\tilde{Z}_i = 0 \quad \forall i = 2, \dots, k. \end{cases} \tag{5.11}$$

Moreover

$$\|\phi\|_2 \le C\|h\|_2, \quad |c_i| \le C\|h\|_2 \quad \forall i = 2, \dots, k. \tag{5.12}$$

Proof. It is analogous to that of Proposition 1 in [2]. □

Lemma 5.4. *Let $0 < \nu < \min(2, \alpha)$ and $\tau > 0$. Then there are $\delta_0 > 0$, C such that for $0 < \delta \le \delta_0$ there is a linear operator T_0 such that given h with $\|h\|_* < \infty$, $(\phi, c_1, \dots, c_k) = T_0(h)$ solves*

$$\phi'' - \alpha^2\phi + 2^* \sum_{i=1}^k U_0(t-\xi_i)^{2^*-1}\phi = h + \sum_{i=1}^k c_i \tilde{Z}_i \quad \text{in } \mathbb{R} \tag{5.13}$$

Moreover

$$\|\phi\|_* \le C\|h\|_* \quad \text{and} \quad |c_i| \le C\|h\|_* \quad \forall i = 1, \dots, k. \tag{5.14}$$

Proof. Define

$$W_i(t) = 2^* U_0(t-\xi_i)^{2^*-1}. \tag{5.15}$$

Let $\eta_1, \eta_2 \in C^\infty(\mathbb{R})$ be such that $0 \le \eta_1, \eta_2 \le 1$, and

$$\begin{cases} \eta_1 \equiv 1 & \text{in } (-\infty, (1+\frac{1}{2\alpha})\log\delta], \quad \eta_1 \equiv 0 \quad \text{in } [(1+\frac{1}{4\alpha})\log\delta, \infty) \\ \eta_2 \equiv 1 & \text{in } (-\infty, (1+\frac{3}{4\alpha})\log\delta], \quad \eta_2 \equiv 0 \quad \text{in } [(1+\frac{1}{2\alpha})\log\delta, \infty). \end{cases}$$

We look for a solution of (5.13) of the form $\phi = \phi_1 + \phi_2\eta_2$. For this it is sufficient that ϕ_1, ϕ_2 satisfy the following system

$$\begin{aligned} \phi_1'' - \alpha^2\phi_1 + W_1\phi_1 =& (1-\eta_2)h + c_1\tilde{Z}_1 - (1-\eta_2)\sum_{i=2}^k W_i\phi_1 \\ & - 2\phi_2'\eta_2' - \phi_2\eta_2'' \end{aligned} \tag{5.16}$$

$$\phi_2'' - \alpha^2\phi_2 + \sum_{i=2}^k W_i\phi_2 = \eta_1 h + \sum_{i=2}^k c_i\tilde{Z}_i - \eta_1 W_1\phi_2 - \eta_1\sum_{i=2}^k W_i\phi_1 \quad \text{in } \mathbb{R}. \tag{5.17}$$

Define the operator $\phi = T_1(h)$ to be the solution of (5.4) of Lemma 5.2 and $\phi = T_2(h)$ to be the solution of (5.11) obtained in Lemma 5.3. Then to find a solution of (5.16), (5.17) with the correct bounds we are led to solve the system

$$\phi_1 = T_1[(1-\eta_2)h - (1-\eta_2)\sum_{i=2}^k W_i\phi_1 - 2\phi_2'\eta_2' - \phi_2\eta_2''] \tag{5.18}$$

$$\phi_2 = T_2[\eta_1 h - \eta_1 W_1\phi_2 - \eta_1\sum_{i=2}^k W_i\phi_1] \tag{5.19}$$

We do this in the Banach space E consisting of pairs (ϕ_1, ϕ_2) of functions $\phi_j : \mathbb{R} \to \mathbb{R}$ such that ϕ_2 is Lipschitz continuous and the following norm is finite

$$\|(\phi_1, \phi_2)\|_E = \|\phi_1\|_1 + \|\phi_2\|_2 + \|\phi_2'\|_2$$

where $\| \ \|_1$ is defined by (5.3) and $\| \ \|_2$ in (5.10). We will verify that the operator $\tilde{T} : E \to E$ defined by the right hand side of (5.18), (5.19) is a contraction on E.

For this we estimate thanks to (5.5)

$$\|T_1[-(1-\eta_2)\sum_{i=2}^k W_i\phi_1 - 2\phi_2'\eta_2' - \phi_2\eta_2'']\|_1$$
$$\le C(\|(1-\eta_2)\sum_{i=2}^k W_i\phi_1\|_1 + \|\phi_2'\eta_2'\|_1 + \|\phi_2\eta_2''\|_1)$$

Some computations show that

$$\|(1-\eta_2)\sum_{i=2}^k W_i\phi_1\|_1 \le C\delta^{\frac{1}{\alpha}}\|\phi_1\|_1$$
$$\|\phi_2'\eta_2'\|_1 \le \frac{C}{|\log\delta|}\|\phi_2'\|_2$$
$$\|\phi_2\eta_2''\|_1 \le \frac{C}{|\log\delta|^2}\|\phi_2\|_2$$

Using (5.12) we have

$$\|T_2[-\eta_1 W_1\phi_2 - \eta_1\sum_{i=2}^k W_i\phi_1]\|_2 \le C(\|\eta_1 W_1\phi_2\|_2 + \|\eta_1\sum_{i=2}^k W_i\phi_1\|_2)$$

By computation we obtain

$$\|\eta_1 W_1\phi_2\|_2 \le C\delta^{\frac{1}{2\alpha}}\|\phi_2\|_2$$

and, if $\nu \ge 1$

$$\|\eta_1\sum_{i=2}^k W_i\phi_1\|_2 \le C\delta^{\frac{3-\nu}{2\alpha}}\|\phi_1\|_1$$

while if $\nu < 1$

$$\|\eta_1\sum_{i=2}^k W_i\phi_1\|_2 \le C\delta^{\frac{\nu}{\alpha}}\|\phi_1\|_1.$$

We see that if $\nu < 3$ then $\tilde{T}$ is a contraction in E. □

Proof of Proposition 5.1. First, let us prove existence of a solution. Let W_i be defined by (5.15). Let us write equation (5.1) in the form

$$\phi = T_0[h + (\sum_{i=1}^k W_i - pe^{\sigma_p t}V^{p-1} - qe^{\sigma_q t}V^{q-1})\phi] \tag{5.20}$$

where T_0 is the operator defined in Lemma 5.4. Let X the Banach space of continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ such that $\|\phi\|_* < \infty$ equipped with the norm $\| \ \|_*$. We observe that that by (5.14) we have

$$\|T_0[(\sum_{i=1}^k W_i - pe^{\sigma_p t}V^{p-1} - qe^{\sigma_q t}V^{q-1})\phi]\|_*$$
$$\le C\|\sum_{i=1}^k W_i - pe^{\sigma_p t}V^{p-1} - qe^{\sigma_q t}V^{q-1}\|_{L^\infty(\mathbb{R})}\|\phi\|_*.$$

A computation shows that

$$\| \sum_{i=1}^{k} W_i - p e^{\sigma_p t} V^{p-1} - q e^{\sigma_q t} V^{q-1} \|_{L^\infty(\mathbb{R})} = o(1) \quad \text{as } \delta \to 0. \tag{5.21}$$

Indeed, let us estimate $\|e^{\sigma_p t} V^{p-1}\|_{L^\infty(\mathbb{R})}$. We have

$$\begin{aligned} e^{\sigma_p t} V^{p-1} = e^{\sigma_p t} \left(\sum_{j=1}^{k} U(t-\xi_j) \right)^{p-1} &\le C e^{\sigma_p t} \left(\sum_{j=1}^{k} e^{\alpha(t-\xi_j)} (1 + e^{2(t-\xi_j)})^{-\beta} \right)^{p-1} \\ &\le C e^{\sigma_p t} \sum_{j=1}^{k} e^{\alpha(p-1)(t-\xi_j)} (1 + e^{2(t-\xi_j)})^{-1}. \end{aligned}$$

For $t \ge \xi_j$

$$e^{\sigma_p t} e^{\alpha(p-1)(t-\xi_j)} (1 + e^{2(t-\xi_j)})^{-1} \le C e^{(\alpha-2\beta)(p-1)(-\xi_j)} \le C e^{(1-\alpha\varepsilon)\xi_j}$$

because $\sigma_p + (\alpha - 2\beta)(p-1) = 0$. Using that ξ satisfy (2.6), (2.7) for some $M > 0$ we find for $t \ge \xi_j$

$$e^{\sigma_p t} e^{\alpha(p-1)(t-\xi_j)} (1 + e^{2(t-\xi_j)})^{-1} \le C \delta^{(1-\alpha\varepsilon)((j-1)/\alpha+1)} \le C \delta^{1-\alpha\varepsilon}.$$

For $t \le \xi_j$

$$\begin{aligned} e^{\sigma_p t} e^{\alpha(p-1)(t-\xi_j)} (1 + e^{2(t-\xi_j)})^{-1} &\le C e^{\sigma_p t} e^{\alpha(p-1)(t-\xi_j)} \le C e^{\sigma_p t} \\ &\le C e^{\sigma_p \xi_j} \le C \delta^{\sigma_p((j-1)/\alpha+1)} \le C \delta^{1-\alpha\varepsilon} \end{aligned}$$

Therefore

$$\| p e^{\sigma_p t} V^{p-1} \|_{L^\infty(\mathbb{R})} \le C \delta^{1-\alpha\varepsilon}.$$

The difference $\sum_{i=1}^{k} W_i - q e^{\sigma_q t} V^{q-1}$ in (5.21) can be handled similarly. This implies that if $\|h\|_* < \infty$ then for $\varepsilon, \delta > 0$ suitably small (5.20) has a unique solution in X. □

6. Proof of Theorem 1.1

Let us fix an integer $k \ge 2$. From Proposition 4.1 there is a function $\varepsilon_k(\Lambda, \delta) > 0$ and $\theta > 1/2$ such that if $\varepsilon = \varepsilon_k(\Lambda, \delta)$ and δ is sufficiently small then $\|E\|_* \le C\delta^\theta$. We claim that for $\delta > 0$ small enough and $\varepsilon = \varepsilon_k(\Lambda, \delta)$ there exists a solution ϕ to the nonlinear projected problem

$$L\phi + E + N(\phi) = \sum_{i=1}^{k} c_i \tilde{Z}_i, \tag{6.1}$$

such that $\|\phi\|_* \le A\delta^\theta$, for a suitable constant $A > 0$. Here $\tilde{Z}_i$ are the functions defined in (5.2). Indeed, let T be the operator defined in Proposition 5.1. Then we obtain a solution of (6.1) if we solve the fixed point problem

$$\phi + T(E - N(\phi)) = 0. \tag{6.2}$$

Let us consider the Banach space X of all continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ with $\|\phi\|_* < +\infty$ with the norm $\| \ \|_*$. Let $A > 0$. Then for $\phi_1, \phi_2 \in E$ with $\|\phi_i\|_* \le A\delta^\theta$, $i = 1, 2$ one can check that

$$\|N(\phi_1) - N(\phi_2)\|_* \le C_A \delta^a \|\phi_1 - \phi_2\|_*$$

for some $a > 0$. We conclude from this estimate and the boundedness of the operator T that the fixed point problem (6.2) has a unique solution ϕ in the region $\|\phi\|_* \le A\delta^\theta$ for some suitably chosen A. We will write this solution as $\phi(\Lambda)$.

To find a solution of (2.9) it remains to verify that one can choose $\Lambda = (\Lambda_1, \dots, \Lambda_k)$ such that the constants c_i in (6.1) are all zero. Testing equation (6.1) against $Z_j(t) = U_0'(t - \xi_j)$ for $i = 1, \dots, k$, we obtain

$$\int_{-\infty}^{\infty} \phi L Z_j + \int_{-\infty}^{\infty} N(\phi) Z_j + \int_{-\infty}^{\infty} E Z_j = c_j \int_{-\infty}^{\infty} \tilde{Z}_j Z_j.$$

Therefore $c_i = 0$ for all i is equivalent to

$$\int_{-\infty}^{\infty} \phi L Z_j + \int_{-\infty}^{\infty} N(\phi) Z_j + \int_{-\infty}^{\infty} E Z_j = 0 \tag{6.3}$$

for all j. A calculation shows that

$$\int_{-\infty}^{\infty} \phi L Z_j + \int_{-\infty}^{\infty} N(\phi) Z_j = o(\delta)$$

as $\delta \to 0$, where $o(\delta)$ a continuous function of Λ that tends to 0 is uniformly in the considered region as $\delta \to 0$ (for this it is important that $\|\phi\|_* \le C\delta^\theta$ with $\theta > 1/2$). Write $\mathcal{E}(v) = v'' - \alpha^2 v + e^{-\sigma_p t} v^p + e^{-\sigma_q t} v^q$. Since $E = \mathcal{E}(V)$ and $Z_i = \partial_{\xi_i} V$ we have that

$$\int_{-\infty}^{\infty} E Z_i = \int_{-\infty}^{\infty} \mathcal{E}(V) \partial_{\xi_i} V = \partial_{\xi_i} I(V)$$

According to the expansion for $I(V)$ in Proposition 3.1 and using the relations (2.6), we see that the system (6.3) is equivalent to

$$\nabla \varphi(\Lambda) + o(1) = 0,$$

where the quantity $o(1)$ goes to zero uniformly on the considered region for the parameters Λ_i and depends continuously on them. We recall that the functional φ possesses a unique critical point Λ^*, which is nondegenerate. Therefore the above equation has a solution that is close to Λ^* for $\delta > 0$ small. □

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References

[1] R. Bamón, I. Flores, M. del Pino, *Ground states of semilinear elliptic equations: a geometric approach.* Ann. Inst. H. Poincaré Anal. Non Linéaire 17 (2000), no. 5, 551–581.

[2] J.F. Campos, *"Bubble-tower" phenomena in a semilinear elliptic equation with mixed Sobolev growth.* Nonlinear Anal. 68 (2008), no. 5, 1382–1397.

[3] J. Dolbeault, I. Flores, *Geometry of phase space and solutions of semilinear elliptic equations in a ball.* Trans. Amer. Math. Soc. 359 (2007), no. 9, 4073–4087.

[4] M. Del Pino, J. Dolbeault, M. Musso, *"Bubble-tower" radial solutions in the slightly supercritical Brezis-Nirenberg problem.* J. Differential Equations 193 (2003), no. 2, 280–306

[5] B. Gidas, W.M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle.* Comm. Math. Phys. 68 (1979), no. 3, 209–243.

[6] B. Gidas, J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations.* Comm. Pure Appl. Math. 34 (1981), no. 4, 525–598

[7] I. Flores, *A resonance phenomenon for ground states of an elliptic equation of Emden-Fowler type.* J. Differential Equations 198 (2004), no. 1, 1–15.

[8] M. del Pino, I. Guerra, *Ground states of a prescribed mean curvature equation.* J. Differential Equations 241 (2007), no. 1, 112–129

[9] C.S. Lin, W.-M. Ni, *A counterexample to the nodal domain conjecture and a related semilinear equation.* Proc. Amer. Math. Soc. 102 (1988), no. 2, 271–277.

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